Oxford School on Neutron Scattering 2024

April 17, 2024

Introductory Mathematics:

Syllabus and preparatory exercises

Syllabus

1. *Trigonometry*
Angles and solid angles

2. *Vectors*
Magnitudes, directions and components; scalar (dot) and vector (cross) products.

3. *Complex numbers*
Definitions; complex conjugate; modulus and argument; $\exp(i\theta)$; representation on Argand diagram.

4. *Calculus*
Derivatives; integrals; delta-functions; Fourier transforms.

Preparatory exercises

1. The O–H bond length in a water molecule is 0.0957 nm, and the H–O–H bond angle is 104.5°. What is the distance between the two H atoms? [Ans: 0.151 nm]

2. What is the solid angle subtended at the earth by (i) the sun, and (ii) the moon? Explain why total solar eclipses can occur.
[The average sun–Earth and moon–Earth distances are 1.50×10^{11} m and 3.84×10^{8} m, respectively, and the radii of the sun and moon are 6.96×10^{8} m and 1.74×10^{6} m, respectively.]
[Ans: 6.76×10^{-5} sr; 6.45×10^{-5} sr]

3. The initial neutron wavevector $\mathbf{k}_{\mathrm{i}}$ has magnitude $k_{\mathrm{i}} = 10\,\mathrm{nm}^{-1}$ and points along the x axis. The final neutron wavevector $\mathbf{k}_{\mathrm{f}}$ has magnitude $k_{\mathrm{f}} = 8\,\mathrm{nm}^{-1}$, and lies in the xy plane at an angle of $\phi = 40°$ to $\mathbf{k}_{\mathrm{i}}$ in an anticlockwise sense when viewed down the z axis. Calculate the magnitude and direction of the scattering vector $\mathbf{Q} = \mathbf{k}_{\mathrm{i}} - \mathbf{k}_{\mathrm{f}}$.
[Ans: $\mathbf{Q}$ has magnitude $Q = 6.44\,\mathrm{nm}^{-1}$ and points at an angle of $-53.0°$ to the x axis.]

4. The position vectors of the points R and S are $\mathbf{r} = (1, 2, 3)$ and $\mathbf{s} = (3, 0, -1)$. Calculate $\mathbf{r} \cdot \mathbf{r}$, $\mathbf{s} \cdot \mathbf{s}$ and $\mathbf{r} \cdot \mathbf{s}$. What does the last result mean?
[Ans: 14, 10, 0. The angle between $\mathbf{r}$ and $\mathbf{s}$ is 90°.]

5. Calculate $\mathbf{r} \times \mathbf{s}$. The volume of a parallelepiped whose sides are defined by the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ is given by $V = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$. Calculate V for the case $\mathbf{a} = (1, 0, 0)$, $\mathbf{b} = (0, 1, 0)$, $\mathbf{c} = (1, 1, 4)$.
[Ans: $(-2, 10, -6)$. $V = 4$. Note that $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a})$ for any $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.]

6. If $u = 2 + 3i$ and $v = 1 - i$, find the real and imaginary parts of (i) $u + v$, (ii) $u - v$, (iii) uv, (iv) u/v.
[Ans: (i) $3 + 2i$, (ii) $1 + 4i$, (iii) $5 + i$, (iv) $-\frac{1}{2} + \frac{5}{2}i$.]

7. If $z = \mathrm{e}^{i\phi} = \cos\phi + i \sin\phi$, find the real and imaginary parts of z when (i) $\phi = 0$, (ii) $\phi = \pi/4$, (iii) $\phi = \pi/3$, (iv) $\phi = \pi/2$, (v) $\phi = 5\pi/4$, (vi) $\phi = 7\pi/3$. Plot these complex numbers on an Argand diagram. Also plot their complex conjugates.
[Ans: (i) 1, (ii) $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$, (iii) $\frac{1}{2} + \frac{\sqrt{3}}{2}i$, (iv) i, (v) $-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$, (vi) $\frac{1}{2} + \frac{\sqrt{3}}{2}i$. The complex conjugate of z is $z^* = \mathrm{e}^{-i\phi} = \cos\phi - i \sin\phi$]

8. Taking u and v from question 6, determine $|u|$ and $|v|$, and express u and v in the form $r\mathrm{e}^{i\phi}$.
[Ans: $\sqrt{13}$ and $\sqrt{2}$. $u = \sqrt{13}\,\mathrm{e}^{0.313\pi}$, $v = \sqrt{2}\,\mathrm{e}^{-\pi/4}$.]

9. Differentiate each of the following functions: (i) $y = Ax^3 + Bx + C$; (ii) $y = 1/\sin x$; (iii) $y = 4\sin^2 kx$; (iv) $y = (x^2 + y^2)/2$.
[Ans: (i) $3Ax^2 + B$; (ii) $-\cos x/\sin^2 x$; (iii) $8k \sin kx \cos kx$; (iv) $\pm x/\sqrt{1 - x^2}$.]

10. Find the maximum value of the function $y = x\mathrm{e}^{-\beta x}$.
[Ans: The maximum value is $1/(\beta \mathrm{e})$ found at $x = 1/\beta$.]

11. Evaluate the following integrals: (i) $\int (1 + 2x + 3x^2)\,\mathrm{d}x$; (ii) $\int x^2 \mathrm{e}^{-x}\,\mathrm{d}x$; (iii) $\int_{-a/2}^{a/2} \mathrm{e}^{-ikx}\,\mathrm{d}x$.
[Ans: (i) $x + x^2 + x^3 + C$; (ii) $-(x^2 + 2x + 2)\mathrm{e}^{-x} + C$; (iii) $(a \sin\phi)/\phi$, where $\phi = ka/2$.]